

ON UNCONVENTIONAL LIMIT SETS OF CONTRACTIVE FUNCTIONS ON \mathbb{Z}_p

FARRUKH MUKHAMEDOV AND OTABEK KHAKIMOV

ABSTRACT. In the present paper, we are going to study metric properties of unconventional limit set of a semigroup G generated by contractive functions $\{f_i\}_{i=1}^N$ on the unit ball \mathbb{Z}_p of p -adic numbers. Namely, we prove that the unconventional limit set is compact, perfect and uniformly disconnected. Moreover, we provide an example of two contractions for which the corresponding unconventional limiting set is quasi-symmetrically equivalent to the symbolic Cantor set.

Mathematics Subject Classification: 46S10, 12J12, 39A70, 47H10, 60K35.

Key words: p -adic numbers, unconventional limit set; compact; uniformly perfect; quasi-symmetric; symbolic Cantor set.

1. INTRODUCTION

Non-Archimedean dynamical systems is one of the most popular areas of the modern mathematics. There are many works devoted to p -adic dynamics [1, 2, 3, 7, 8, 10, 14, 15, 17, 22, 24]. On the other hand, in investigations of random mappings it has been studied the metric properties of limit sets in the Euclidean spaces (see for example [5, 4, 12, 13]). These investigations have found their applications in the fractal geometry [6, 11]. This naturally motivates to consider the metric properties of limit sets in a non-Archimedean setting. In this direction, very recently, in [20] it has been considered a semigroup G generated by a finite set $\{f_i\}_{i=1}^N$ of contractive functions on \mathcal{O} (here $\mathcal{O} = \{x \in \mathbb{K} : |x| \leq 1\}$ is the closed unit ball of the non-Archimedean field \mathbb{K}). Namely, $G = \bigcup_{k \geq 1} G_k$, where $G_k = \{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}, 1 \leq i_j \leq N, 1 \leq j \leq k\}$. Furthermore, it was studied metric properties of the limit set Λ of G which is a complement of the set of all points $x \in \mathcal{O}$ for which there exist open neighborhoods U_x of x such that $g(U_x) \cap U_x = \emptyset$ for all but finitely many $g \in G$. Note that the limit set Λ of G is a very important object in the study of random dynamical systems (see for example [8, 23, 24]).

It is known that the composition of two contractive mappings is also contraction. But, in general, the product or the sum of such kind of contractions is contraction. But in a non-Archimedean case they are also contractions. Using that fact in [19] it has been studied the uniqueness limiting set of unconventional iterates of contractive mappings. Note that these results play an important role in the theory of p -adic Gibbs measures [9], [16]-[18].

In the present paper, we are going to study metric properties of unconventional limit set of contractive function on \mathbb{Z}_p . Namely, we prove that the unconventional limit set is compact, perfect and uniformly disconnected. Moreover, we provide an example of two contractions for which the corresponding unconventional limiting set is quasi-symmetrically equivalent to the symbolic Cantor set. This paper can be considered as a generalization and extension of some results of [20].

2. DEFINITIONS AND PRELIMINARY RESULTS

A metric space X is said to be *doubling* if there is a constant k such that every disk B in X can be covered with at most k disks of half the radius of B . A number of metric spaces have this property, e.g. the Euclidean space, a compact Riemann surface, etc. However a non-Archimedean space is not necessary a doubling space, e.g. \mathbb{C}_p . Hence it is very important to know whether a subspace of a non-Archimedean space is a doubling space. Note that \mathbb{Q}_p is a doubling. Let (X, d) be a complete metric space. Let \mathcal{B} be the collection of all bounded subsets of X . For a set $E \in \mathcal{B}$, we denote the diameter of E by $\text{diam}(E) = \sup_{z, w \in E} d(z, w)$. By definition, a set $E \in \mathcal{B}$ is called a *uniformly perfect set* if E contains at least two points and there exists a constant $c > 0$ such that for any point $x_0 \in E$ and $0 < r < \text{diam}(E)$, the annulus $\{x \in X : cr \leq d(x, x_0) \leq r\}$ meets E . We say that a metric space (X, d) is *uniformly disconnected* if there is a constant $C > 1$ so that for each $x \in X$ and $r > 0$ we can find a closed subset A of X such that $B_{r/C}(x) \subset A \subset B_r(x)$, and $\text{dist}(A, X \setminus A) \geq C^{-1}r$. Let $(X, d_X), (Y, d_Y)$ be metric spaces. A mapping $f : X \rightarrow Y$ is said to be *quasi-symmetric* if it is not constant and if there exists a homeomorphism $\eta : [0, +\infty) \rightarrow [0, +\infty)$ such that for any points $x, y, z \in X$ and $t > 0$ we have that $d_X(x, y) \leq td_X(x, z)$ implies that $d_Y(f(x), f(y)) \leq \eta(t)d_Y(f(x), f(z))$. Two metric spaces are said to be *quasi-symmetrically equivalent* if there is a quasi-symmetric mapping from one to the other.

Let $F = \{0, 1\}$ and F^∞ denote the set of sequences $\{x_i\}_{i=1}^\infty$ with $x_i \in F$ for each i . Given $x = \{x_i\}$ and $y = \{y_i\}$ in F^∞ , let $L(x, y)$ be the largest integer l such that $x_i = y_i$ when $1 \leq i \leq l$, and set $L(x, y) = \infty$ when $x = y$. Take $0 < a < 1$ and set $d_a(x, y) = a^{L(x, y)}$. Then F^∞ endowed with the metric $d_a(x, y)$ is a standard symbolic Cantor set. The space Z_2 of 2-adic integers can be regarded as the standard symbolic Cantor set F^∞ endowed with the metric $d_{1/2}(x, y)$.

Theorem 2.1. [11] *Suppose that (X, d) is a compact metric space which is bounded, complete, doubling, uniformly disconnected, and uniformly perfect. Then X is quasi-symmetrically equivalent to the symbolic Cantor set F^∞ , where we take $F = \{0, 1\}$ with the metric $d_a(x, y) = a^{L(x, y)}$ where $a = 1/2$.*

2.1. p -adic numbers. In what follows p will be a fixed prime number. The set \mathbb{Q}_p is defined as a completion of the rational numbers \mathbb{Q} with respect to the norm $|\cdot|_p : \mathbb{Q} \rightarrow \mathbb{R}_+$ given by

$$(2.1) \quad |x|_p = \begin{cases} p^{-r} & x \neq 0, \\ 0, & x = 0, \end{cases}$$

here, $x = p^r \frac{m}{n}$ with $r, m \in \mathbb{Z}, n \in \mathbb{N}, (m, p) = (n, p) = 1$. The absolute value $|\cdot|_p$ is non-Archimedean, meaning that it satisfies the strong triangle inequality $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. We recall a nice property of the norm, i.e. if $|x|_p > |y|_p$ then $|x + y|_p = |x|_p$. Note that this is a crucial property which is proper to the non-Archimedeanity of the norm.

Any p -adic number $x \in \mathbb{Q}_p, x \neq 0$ can be uniquely represented in the form

$$(2.2) \quad x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p - 1, x_0 > 0, j = 0, 1, 2, \dots$. In this case $|x|_p = p^{-\gamma(x)}$. Note that the basics of p -adic analysis are given in [21].

The next lemma immediately follows from the properties of the p -adic norm

Lemma 2.2. *Let f be a mapping on \mathbb{Q}_p . Then the following statements are equivalent*

- (a) f is contraction;
- (b) $|f(x) - f(y)|_p < |x - y|_p$ for all $x, y \in \mathbb{Q}_p$;
- (c) $|f(x) - f(y)|_p \leq \frac{1}{p}|x - y|_p$ for all $x, y \in \mathbb{Q}_p$.

For each $a \in \mathbb{Q}_p$, $r > 0$ we denote

$$B_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}, \quad B_r^-(a) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$$

and the set of all p -adic integers $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

3. UNCONVENTIONAL LIMIT SET

Let M, N, L be fixed positive integers. For convenience, instead of $\{1, 2, \dots, N\}$ we will write $[1, N]$. Let consider a family $\xi := \{\xi_{ij} : [1, N] \rightarrow [1, N] : (i, j) \in [1, M] \times [1, L]\}$ of mappings. Let $\{f_i\}_{i=1}^N$ be a finite set of contractive mappings on \mathbb{Z}_p . In what follows, we frequently use the denotation $\Sigma = [1, N]^{\mathbb{N}}$.

For each $\alpha \in \Sigma$, $n \in \mathbb{N}$ we denote

$$(3.1) \quad F_{\alpha, n} = \sum_{i=1}^M \prod_{j=1}^L F_{\alpha, n}^{\xi_{ij}}.$$

where

$$(3.2) \quad F_{\alpha, n}^{\xi_{ij}} = f_{\xi_{ij}(\alpha_1)} \circ \dots \circ f_{\xi_{ij}(\alpha_n)}, \quad \alpha = (\alpha_1, \dots, \alpha_n, \dots).$$

Put

$$\mathcal{F}_\xi = \bigcup_{n \geq 1} \mathcal{F}_{\xi, n}, \quad \mathcal{F}_{\xi, n} = \{F_{\alpha, n} : \alpha \in \Sigma\}.$$

The set \mathcal{F}_ξ is called a *unconventional set* of the semigroup G generated by a finite set $\{f_i\}_{i=1}^N$.

Remark 3.1. In the sequel, we always assume that the family ξ satisfies the following condition

$$(3.3) \quad \bigcup_{k=1}^N \bigcup_{i=1}^M \bigcup_{j=1}^L \{\xi_{ij}(k)\} = [1, N].$$

Otherwise, the set \mathcal{F}_ξ will be generated by a subset of $\{f_i\}_{i=1}^N$.

Example 3.1. Let us construct an example of a family of mappings $\xi = \{\xi_{ij} : [1, N] \rightarrow [1, N]\}$ which satisfies (3.3). For any integer number $\ell \in [1, N]$ we define an action on $[1, N]$ by

$$(\ell * k) = \begin{cases} (\ell + k) \pmod{N} & \text{if } N \nmid (\ell + k) \\ N & \text{if } N \mid (\ell + k). \end{cases} \quad k \in [1, N].$$

Then for any number $\xi_{ij} \in [1, N]$ we define $\xi_{ij}(k) := (\xi_{ij} * k)$, $k \in [1, N]$. It is clear that $\xi_{ij} : [1, N] \rightarrow [1, N]$ and (3.3) holds.

Lemma 3.1. *For any $F \in \mathcal{F}_\xi$ the following statements hold:*

- (a) $F : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$;
- (b) F is a contraction.

Proof. Let $F \in \mathcal{F}_\xi$. By construction of the set \mathcal{F}_ξ there exist $\alpha \in \Sigma$ and an integer number $n \geq 1$ such that

$$(3.4) \quad F = \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}}.$$

(a) Since $f_k : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, $k = \overline{1, N}$ by (3.2) we have $F_{\alpha,n}^{\xi_{ij}} : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ for any $(i, j) \in [1, M] \times [1, L]$. Consequently, for any $x \in \mathbb{Z}_p$ using the strong triangle inequality from (3.4) one can find

$$|F(x)|_p \leq \max_{1 \leq i \leq M} \left| \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}}(x) \right|_p \leq \max_{\substack{1 \leq i \leq M \\ 1 \leq j \leq L}} |F_{\alpha,n}^{\xi_{ij}}(x)|_p \leq 1.$$

(b) Let $x, y \in \mathbb{Z}_p$. Then we have

$$(3.5) \quad \begin{aligned} F(x) - F(y) &= \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}}(x) - \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}}(y) \\ &= \sum_{i=1}^M \sum_{j=1}^L (F_{\alpha,n}^{\xi_{ij}}(x) - F_{\alpha,n}^{\xi_{ij}}(y)) \prod_{k>j} F_{\alpha,n}^{\xi_{ik}}(x) \prod_{l<j} F_{\alpha,n}^{\xi_{il}}(y) \end{aligned}$$

Again by means of the strong triangle inequality from (3.5), one gets

$$(3.6) \quad |F(x) - F(y)|_p \leq \max_{\substack{1 \leq i \leq M \\ 1 \leq j \leq L}} |F_{\alpha,n}^{\xi_{ij}}(x) - F_{\alpha,n}^{\xi_{ij}}(y)|_p$$

By (3.2) we have

$$|F_{\alpha,n}^{\xi_{ij}}(x) - F_{\alpha,n}^{\xi_{ij}}(y)|_p \leq \frac{1}{p} |x - y|_p \quad \text{for any } (i, j) \in [1, M] \times [1, L].$$

Now substituting the last one into (3.6) one finds the required assertion. \square

A main aim of this paper is to study limiting set of \mathcal{F}_ξ . By *unconventional limit set* of \mathcal{F}_ξ is meant the set Λ^ξ which the compliment of the *discontinuity set* Ω^ξ , i.e. $\mathbb{Z}_p \setminus \Omega^\xi$. The discontinuity set $\Omega^\xi \subset \mathbb{Z}_p$ of \mathcal{F}_ξ is defined as follows: $x \in \Omega^\xi$ if and only if there is a disk $B_r(x)$ such that there are only finitely many $F \in \mathcal{F}_\xi$ satisfying $F(B_r(x)) \cap B_r(x) = \emptyset$.

4. SOME PROPERTIES OF THE UNCONVENTIONAL LIMIT SET

In this section we study several metric properties of the set Λ^ξ .

Denote

$$(4.1) \quad \Lambda_0^\xi = \left\{ \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha}^{(n)} : x_{\xi_{ij}, \alpha}^{(n)} = F_{\alpha,n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha}^{(n)}) \text{ for some } \alpha \in \Sigma \text{ and } n \in \mathbb{N} \right\}.$$

Theorem 4.1. *The limit set Λ^ξ of \mathcal{F}_ξ coincides with the closure of Λ_0^ξ .*

Proof. Let us first show that Λ^ξ is closed. It is enough to establish that Ω^ξ is open. Take any $x \in \Omega^\xi$. Then there exist $r > 0$ and $\{F_k\}_{k=1}^m \subset \mathcal{F}_\xi$ such that $F_k(B_r(x)) \cap B_r(x) = \emptyset$, $k = \overline{1, m}$. Since for any $y \in B_r^-(x)$, one has $B_r(y) = B_r(x)$, hence we have $F_k(B_r(y)) \cap B_r(y) = \emptyset$. This implies that $B_r^-(x) \subset \Omega^\xi$, so Ω^ξ is open.

Let $x \in \Lambda_0^\xi$. Then there exist $\alpha \in \Sigma$ and an integer number $n \geq 1$ such that

$$(4.2) \quad x = \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha}^{(n)},$$

where $x_{\xi_{ij}, \alpha}^{(n)}$ is a fixed point of $F_{\alpha, n}^{\xi_{ij}}$.

Consider a sequence $\{F_m\}_{m=1}^\infty$ defined by

$$(4.3) \quad F_m = \sum_{i=1}^M \prod_{j=1}^L (F_{\alpha, n}^{\xi_{ij}})^m = \sum_{i=1}^M \prod_{j=1}^L (f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)})^m.$$

It is clear that for any $m \geq 1$, we have $F_m \in \mathcal{F}_{\xi, nm}$, hence $F_m \in \mathcal{F}_\xi$. Take any $r > 0$ and $y \in B_r(x)$. Then from (4.2) and (4.3) one finds

$$\begin{aligned} F_m(y) - x &= \sum_{i=1}^M \prod_{j=1}^L (F_{\alpha, n}^{\xi_{ij}})^m(y) - \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha}^{(n)} \\ &= \sum_{i=1}^M \prod_{j=1}^L (F_{\alpha, n}^{\xi_{ij}})^m(y) - \sum_{i=1}^M \prod_{j=1}^L F_{\alpha, n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha}^{(n)}) \\ &= \sum_{i=1}^M \prod_{j=1}^L (F_{\alpha, n}^{\xi_{ij}})^m(y) - \sum_{i=1}^M \prod_{j=1}^L (F_{\alpha, n}^{\xi_{ij}})^m(x_{\xi_{ij}, \alpha}^{(n)}) \\ &= \sum_{i=1}^M \sum_{j=1}^L \left[(F_{\alpha, n}^{\xi_{ij}})^m(y) - (F_{\alpha, n}^{\xi_{ij}})^m(x_{\xi_{ij}, \alpha}^{(n)}) \right] \prod_{k>j} (F_{\alpha, n}^{\xi_{ik}})^m(y) \prod_{l<j} (F_{\alpha, n}^{\xi_{il}})^m(x_{\xi_{il}, \alpha}^{(n)}). \end{aligned}$$

The last equality with the strong triangle inequality implies that

$$(4.4) \quad |F_m(y) - x|_p \leq \max_{i,j} \left| (F_{\alpha, n}^{\xi_{ij}})^m(y) - (F_{\alpha, n}^{\xi_{ij}})^m(x_{\xi_{ij}, \alpha}^{(n)}) \right|_p.$$

The contractivity $F_{\alpha, n}^{\xi_{ij}}$ with (4.4) yields

$$|F_m(y) - x|_p \leq \frac{1}{p^m} \max_{i,j} |y - x_{\xi_{ij}, \alpha}^{(n)}|_p \leq \frac{1}{p^m}.$$

Then there exists a positive integer number m_r such that $F_m(y) \in B_r(x)$ for all $m > m_r$. This means that $F^m(B_r(x)) \cap B_r(x) \neq \emptyset$. Consequently, $x \in \Lambda^\xi$. Since Λ^ξ is closed, we have $\overline{\Lambda}_0^\xi \subset \Lambda^\xi$.

Now suppose that $x_0 \notin \overline{\Lambda}_0^\xi$. Then there exists $r > 0$ such that $B_r(x_0) \cap \overline{\Lambda}_0^\xi = \emptyset$. Choose a positive integer n_0 such that $\frac{1}{p^{n_0}} < r$. Consider a function $F \in \mathcal{F}_\xi$ defined by

$$F = \sum_{i=1}^M \prod_{j=1}^L F_{\alpha, n_0}^{\xi_{ij}} \quad \text{for some } \alpha \in \Sigma.$$

It is easy to see that

$$(4.5) \quad |F(x) - F(y)|_p \leq \frac{1}{p^n} |x - y|_p \leq \frac{1}{p^n} \quad \text{for any } x, y \in \mathbb{Z}_p.$$

Denote

$$x_F = \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha}^{(n_0)},$$

here as before $x_{\xi_{ij},\alpha}^{(n_0)}$ is a fixed point of $F_{\alpha,n_0}^{\xi_{ij}}$. Then $x_F \in \Lambda_0^\xi$. According to Lemma 3.1 the function F has a unique fixed point z_F on \mathbb{Z}_p . Now from the strong triangle inequality and (4.5) we obtain

$$\begin{aligned}
|z_F - x_F|_p &= |F(z_F) - x_F|_p \\
&= \left| \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n_0}^{\xi_{ij}}(z_F) - \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n_0}^{\xi_{ij}}(x_{\xi_{ij},\alpha}^{(n_0)}) \right|_p \\
&= \left| \sum_{i=1}^M \sum_{j=1}^L \left[F_{\alpha,n_0}^{\xi_{ij}}(z_F) - F_{\alpha,n_0}^{\xi_{ij}}(x_{\xi_{ij},\alpha}^{(n_0)}) \right] \prod_{\substack{k>j \\ l<j}} F_{\alpha,n_0}^{\xi_{ik}}(z_F) F_{\alpha,n_0}^{\xi_{il}}(x_{\xi_{il},\alpha}^{(n_0)}) \right|_p \\
(4.6) \quad &\leq \max_{i,j} \left| F_{\alpha,n_0}^{\xi_{ij}}(z_F) - F_{\alpha,n_0}^{\xi_{ij}}(x_{\xi_{ij},\alpha}^{(n_0)}) \right|_p \prod_{\substack{k>j \\ l<j}} \left| F_{\alpha,n_0}^{\xi_{ik}}(z_F) F_{\alpha,n_0}^{\xi_{il}}(x_{\xi_{il},\alpha}^{(n_0)}) \right|_p \leq \frac{1}{p^{n_0}} < r.
\end{aligned}$$

For any $y \in B_r^-(x_0)$ due to contractivity of F one gets

$$|F(y) - z_F|_p = |F(y) - F(z_F)|_p < |y - z_F|_p.$$

Again the strong triangle inequality implies

$$|F(y) - y|_p = |F(y) - z_F + z_F - y|_p = |y - z_F|_p.$$

Since $y \in B_r^-(x_0)$ and $B_r^-(x_0) \cap \overline{\Lambda}_0^\xi = \emptyset$, $x_F \in \Lambda_0^\xi$ one concludes that $|y - x_F|_p > r$. Therefore, from (4.6) it follows that

$$|y - z_F|_p = |y - x_F + x_F - z_F|_p = |y - x_F|_p > r.$$

Hence,

$$|F(y) - y|_p = |y - x_F|_p > r.$$

Consequently, with $|x_0 - y|_p < r$ one finds

$$|F(y) - x_0|_p = |F(y) - y + y - x_0|_p = |F(y) - y|_p > r.$$

This means that $F(y)$ does not belong to the disk $B_r(x_0)$. Hence, $F(B_r(x_0)) \cap B_r(x_0) = \emptyset$ which implies $x_0 \in \Omega^\xi$. It follows that $\Lambda^\xi \subset \overline{\Lambda}_0^\xi$. Hence $\Lambda^\xi = \overline{\Lambda}_0^\xi$. This completes the proof. \square

Now we need the following auxiliary fact.

Lemma 4.2. *Let $\alpha \in \Sigma$. Then a sequence defined by*

$$(4.7) \quad x_\alpha^{(n)} = \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij},\alpha}^{(n)}$$

converges as $n \rightarrow \infty$. Here as before $x_{\xi_{ij},\alpha}^{(n)}$ be a fixed point of $F_{\alpha,n}^{\xi_{ij}}$.

Proof. Due to the closedness of \mathbb{Z}_p it is enough to show that the sequence (4.7) is Cauchy.

Take an arbitrary $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\frac{1}{p^{n_0}} < \varepsilon$. Then for any $n, m \geq n_0$ ($n > m$) we have

$$\begin{aligned} |x_\alpha^{(n)} - x_\alpha^{(m)}|_p &= \left| \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha}^{(n)} - \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha}^{(m)} \right|_p \\ &\leq \max_{i,j} \left| x_{\xi_{ij}, \alpha}^{(n)} - x_{\xi_{ij}, \alpha}^{(m)} \right|_p = \max_{i,j} \left| F_{\alpha, n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha}^{(n)}) - F_{\alpha, m}^{\xi_{ij}}(x_{\xi_{ij}, \alpha}^{(m)}) \right|_p \\ &\leq \frac{1}{p^m} \max_{i,j} \left| (f_{\xi_{ij}(\alpha_{m+1})} \circ \dots \circ f_{\xi_{ij}(\alpha_n)}) (x_{\xi_{ij}, \alpha}^{(n)} - x_{\xi_{ij}, \alpha}^{(m)}) \right|_p \leq \frac{1}{p^m} < \varepsilon. \end{aligned}$$

This means that $\{x_\alpha^{(n)}\}$ is a Cauchy sequence. □

For a given $\alpha \in \Sigma$ due to Lemma 4.2 we denote

$$x_\alpha = \lim_{n \rightarrow \infty} x_\alpha^{(n)}.$$

Put

$$\tilde{\Lambda}^\xi = \{x_\alpha : \alpha \in \Sigma\}.$$

Theorem 4.3. *One has $\Lambda^\xi = \tilde{\Lambda}^\xi$. Moreover, Λ^ξ is compact.*

Proof. Let us first show that $\tilde{\Lambda}^\xi$ is compact. To do so, we define a mapping $\pi : \Sigma \rightarrow \tilde{\Lambda}^\xi$ as follows:

$$\pi(\alpha) = x_\alpha.$$

It is known that Σ is compact and to establish compactness of $\tilde{\Lambda}^\xi$ it is sufficient to show that π is continuous. Suppose that $\alpha = (\alpha_1, \dots, \alpha_k, \dots) \in \Sigma$ and $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\frac{1}{p^{n_0}} < \varepsilon$. Then for any $\beta \in \Sigma$ with $\beta_k = \alpha_k$, $k \leq n_0$, we find

$$\begin{aligned} (4.8) \quad & \left| \sum_{i=1}^M \prod_{j=1}^L F_{\beta, n_0}^{\xi_{ij}}(x_{\xi_{ij}, \beta}^{(n_0)}) - \sum_{i=1}^M \prod_{j=1}^L F_{\alpha, n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha}^{(n)}) \right|_p \\ &= \left| \sum_i \sum_j \left[F_{\beta, n_0}^{\xi_{ij}}(x_{\xi_{ij}, \beta}^{(n_0)}) - F_{\alpha, n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha}^{(n)}) \right] \prod_{\substack{k > j \\ l < j}} F_{\beta, n_0}^{\xi_{ik}}(x_{\xi_{ik}, \beta}^{(n_0)}) F_{\alpha, n}^{\xi_{il}}(x_{\xi_{il}, \alpha}^{(n)}) \right|_p \\ &\leq \max_{i,j} \left| F_{\beta, n_0}^{\xi_{ij}}(x_{\xi_{ij}, \beta}^{(n_0)}) - F_{\alpha, n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha}^{(n)}) \right|_p \leq \frac{1}{p^{n_0}} \max_{i,j} \left| x_{\xi_{ij}, \beta}^{(n_0)} - x_{\xi_{ij}, \alpha}^{(n)} \right|_p \leq \frac{1}{p^{n_0}} < \varepsilon. \end{aligned}$$

From (4.8) as $n \rightarrow \infty$ one gets

$$|x_\beta^{(n_0)} - x_\alpha|_p < \varepsilon.$$

It follows that π is continuous. Hence, $\tilde{\Lambda}^\xi$ is compact. It is clear that $\Lambda_0^\xi \subset \tilde{\Lambda}^\xi \subset \Lambda^\xi$. Therefore, due to closedness of $\tilde{\Lambda}^\xi$ and $\overline{\Lambda}_0^\xi = \Lambda^\xi$ we immediately find $\Lambda^\xi = \tilde{\Lambda}^\xi$. Consequently, Λ is compact. The proof is complete. □

Lemma 4.4. *Let $F = \sum_{i=1}^M \prod_{j=1}^L F_{\alpha, n}^{\xi_{ij}} \in \mathcal{F}_\xi$. For any $x = \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \beta}^{(\ell)} \in \Lambda_0^\xi$ let us define a mapping by*

$$(4.9) \quad \tilde{F}[x] := \sum_{i=1}^M \prod_{j=1}^L F_{\alpha, n}^{\xi_{ij}}(x_{\xi_{ij}, \beta}^{(\ell)}),$$

Then one has $\tilde{F}[\Lambda_0^\xi] \subset \Lambda^\xi$.

Proof. Let us establish $\tilde{F}[x] \in \Lambda^\xi$. Take any $r > 0$ and $y \in B_r(F[x])$. Consider the following sequence

$$F_m = \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}} \circ \left(F_{\beta,l}^{\xi_{ij}} \right)^m.$$

It is clear that $F_m \in \mathcal{F}_\xi$ for all $m \geq 1$. Then one gets

$$\begin{aligned} \left| F_m(y) - \tilde{F}[x] \right|_p &= \left| \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}} \circ \left(F_{\beta,l}^{\xi_{ij}} \right)^m (y) - \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(\ell)}) \right|_p \\ &\leq \max_{i,j} \left| F_{\alpha,n}^{\xi_{ij}} \circ \left(F_{\beta,l}^{\xi_{ij}} \right)^m (y) - F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(\ell)}) \right|_p \\ &\leq \frac{1}{p^n} \max_{i,j} \left| (F_{\beta,l}^{\xi_{ij}})^m (y) - x_{\xi_{ij},\beta}^{(\ell)} \right|_p \\ &< \max_{i,j} \left| (F_{\beta,l}^{\xi_{ij}})^m (y) - (F_{\beta,l}^{\xi_{ij}})^m (x_{\xi_{ij},\beta}^{(\ell)}) \right|_p \leq \frac{1}{p^m} \max_{i,j} \left| y - x_{\xi_{ij},\beta}^{(\ell)} \right|_p \leq \frac{1}{p^m} \end{aligned}$$

This means that there exists $m_r \in \mathbb{N}$ such that

$$F_m(B_r(F[x])) \cap B_r(x) \neq \emptyset \quad \text{for all } m \geq m_r$$

which yields that $\tilde{F}[x] \in \Lambda^\xi$. □

Remark 4.1. Since $\Lambda_0^\xi \subset \Lambda^\xi$ a natural question arises: how can we extend the function (4.9) to Λ^ξ ?

Given $\alpha, \beta \in \Sigma$ and $n \in \mathbb{N}$ we define an element of Σ by

$$\alpha^{[n]} \vee \beta = (\alpha_1, \dots, \alpha_n, \beta_1, \beta_2, \dots)$$

Lemma 4.5. *For each $F_{\alpha,n} \in \mathcal{F}_{\xi,n}$ and $\beta \in \Sigma$ the sequence*

$$\left\{ \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(m)}) \right\}_{m \in \mathbb{N}}$$

is Cauchy. Here, as before, $x_{\xi_{ij},\beta}^{(m)}$ is a fixed point of $F_{\beta,m}^{\xi_{ij}}$.

Proof. The proof immediately follows from

$$\begin{aligned} \left| \sum_i \prod_j F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(m)}) - \sum_i \prod_j F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(l)}) \right|_p &\leq \max_{i,j} \left| F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(m)}) - F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(l)}) \right|_p \\ &\leq \frac{1}{p^n} \max_{i,j} \left| x_{\xi_{ij},\beta}^{(m)} - x_{\xi_{ij},\beta}^{(l)} \right|_p \\ &\leq \frac{1}{p^{n+\min\{m,l\}}} \rightarrow 0 \end{aligned}$$

□

For any $F = \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}} \in \mathcal{F}_\xi$ we define

$$(4.10) \quad \tilde{F}[x_\beta] := \lim_{m \rightarrow \infty} \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}} (x_{\xi_{ij},\beta}^{(m)})$$

Theorem 4.6. For any $F = \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}} \in \mathcal{F}_\xi$ one has

$$\tilde{F}[x_\beta] = x_{\alpha^{[n]}\vee\beta}.$$

Moreover, $\tilde{F}[\Lambda^\xi] \subset \Lambda^\xi$.

Proof. By definition we have

$$(4.11) \quad x_{\alpha^{[n]}\vee\beta} = \lim_{m \rightarrow \infty} \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha^{[n]}\vee\beta}^{(m)},$$

where $x_{\xi_{ij}, \alpha^{[n]}\vee\beta}^{(m)}$ is a fixed point of

$$\underbrace{f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)}}_{F_{\alpha,n}^{\xi_{ij}}} \circ \underbrace{f_{\xi_{ij}(\beta_1)} \circ \cdots \circ f_{\xi_{ij}(\beta_{m-n})}}_{F_{\beta,m-n}^{\xi_{ij}}}$$

On the other hand, one has

$$(4.12) \quad \tilde{F}[x_\beta] = \lim_{m \rightarrow \infty} \sum_{i=1}^M \prod_{j=1}^L F_{\alpha,n}^{\xi_{ij}}(x_{\xi_{ij},\beta}^{(m)})$$

Hence, one gets

$$\begin{aligned} \left| \sum_i \prod_j F_{\alpha,n}^{\xi_{ij}}(x_{\xi_{ij},\beta}^{(m)}) - \sum_i \prod_j x_{\xi_{ij}, \alpha^{[n]}\vee\beta}^{(m)} \right|_p &\leq \max_{i,j} \left| F_{\alpha,n}^{\xi_{ij}}(x_{\xi_{ij},\beta}^{(m)}) - x_{\xi_{ij}, \alpha^{[n]}\vee\beta}^{(m)} \right|_p \\ &= \max_{i,j} \left| F_{\alpha,n}^{\xi_{ij}}(x_{\xi_{ij},\beta}^{(m)}) - F_{\alpha,n}^{\xi_{ij}} \circ F_{\beta,m-n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha^{[n]}\vee\beta}^{(m)}) \right|_p \\ &\leq \frac{1}{p^n} \max_{i,j} \left| F_{\beta,m}^{\xi_{ij}}(x_{\xi_{ij},\beta}^{(m)}) - F_{\beta,m-n}^{\xi_{ij}}(x_{\xi_{ij}, \alpha^{[n]}\vee\beta}^{(m)}) \right|_p \\ &\leq \frac{1}{p^m} \end{aligned}$$

Hence, from (4.11) and (4.12) we find the desired equality. This completes the proof. \square

Theorem 4.7. If Λ^ξ contains at least two points then it is perfect.

Proof. Let Λ^ξ contain at least two points. Since $\Lambda^\xi = \tilde{\Lambda}^\xi$ and $\bar{\Lambda}_0^\xi = \tilde{\Lambda}^\xi$ it is enough to show that each $x \in \Lambda_0^\xi$ is not isolated point of $\tilde{\Lambda}^\xi$.

Let $x \in \Lambda_0^\xi$. Then there exist $\alpha \in \Sigma$ and a positive integer n such that

$$x = \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij}, \alpha}^{(n)},$$

where $x_{\xi_{ij}, \alpha}^{(n)}$ is a fixed point of $F_{\alpha,n}^{\xi_{ij}}$. It is clear that

$$x = \sum_{i=1}^M \prod_{j=1}^L (F_{\alpha,n}^{\xi_{ij}})^m(x_{\xi_{ij}, \alpha}^{(n)}) \quad \text{for all } m \geq 1$$

Take any $r > 0$ and $y_\beta \in \tilde{\Lambda}^\xi \setminus B_r(x)$. Choose a positive integer m such that $\frac{1}{p^m} < \frac{r}{2}$. Take $\gamma \in \Sigma$ such that $\gamma_{jn+i} = \alpha_i$ for all $i = \overline{1, n}$ and $j = \overline{0, m-1}$.

Note that

$$F_{\gamma,mn}^{\xi_{ij}} = \underbrace{(f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)}) \circ \cdots \circ (f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)})}_m = (F_{\alpha,n}^{\xi_{ij}})^m$$

So,

$$F = \sum_{i=1}^M \prod_{j=1}^L F_{\gamma,mn}^{\xi_{ij}} \in \mathcal{F}_\xi.$$

Due to Theorem 4.6 we have

$$\tilde{F}[y_\beta] = \lim_{k \rightarrow \infty} \sum_{i=1}^M \prod_{j=1}^L F_{\gamma,mn}^{\xi_{ij}}(y_{\xi_{ij},\beta}^{(k)})$$

and belongs to $\tilde{\Lambda}^\xi$. Now choose $k \geq 1$ such that

$$(4.13) \quad \left| \tilde{F}[y_\beta] - \sum_{i=1}^M \prod_{j=1}^L F_{\gamma,mn}^{\xi_{ij}}(y_{\xi_{ij},\beta}^{(k)}) \right|_p < \frac{r}{2}.$$

One can see that

$$\begin{aligned} \tilde{F}[y_\beta] - x &= \tilde{F}[y_\beta] - \sum_{i=1}^M \prod_{j=1}^L F_{\gamma,mn}^{\xi_{ij}}(y_{\xi_{ij},\beta}^{(k)}) + \sum_{i=1}^M \prod_{j=1}^L F_{\gamma,mn}^{\xi_{ij}}(y_{\xi_{ij},\beta}^{(k)}) \\ &\quad - \sum_{i=1}^M \prod_{j=1}^L (F_{\alpha,n}^{\xi_{ij}})^m(x_{\xi_{ij},\alpha}^{(n)}) \\ &= \tilde{F}[y_\beta] - \sum_i \prod_j F_{\gamma,mn}^{\xi_{ij}}(y_{\xi_{ij},\beta}^{(k)}) \\ &\quad + \sum_i \sum_j \left[F_{\gamma,mn}^{\xi_{ij}}(y_{\xi_{ij},\beta}^{(k)}) - F_{\gamma,mn}^{\xi_{ij}}(x_{\xi_{ij},\alpha}^{(n)}) \right] \prod_{\substack{l>j \\ u<j}} F_{\gamma,mn}^{\xi_{il}}(y_{\xi_{il},\beta}^{(k)}) F_{\gamma,mn}^{\xi_{iu}}(x_{\xi_{iu},\alpha}^{(n)}) \end{aligned} \quad (4.14)$$

Noting

$$\left| F_{\gamma,mn}^{\xi_{ij}}(y_{\xi_{ij},\beta}^{(k)}) - F_{\gamma,mn}^{\xi_{ij}}(x_{\xi_{ij},\alpha}^{(n)}) \right|_p \leq \frac{1}{p^{mn}} < \frac{1}{p^m} < \frac{r}{2}.$$

and using (4.13) and the strong triangle inequality from (4.14) we obtain

$$\left| \tilde{F}[y_\beta] - x \right|_p < r$$

Hence, $\tilde{F}[y_\beta] \in \tilde{\Lambda}^\xi \cap B_r(x)$. This means that x is not isolated point of $\tilde{\Lambda}^\xi$. Consequently, Λ^ξ is perfect. \square

Theorem 4.8. *The limiting set Λ^ξ is doubling and uniformly disconnected.*

Proof. Note that \mathbb{Q}_p is a doubling. Since $\Lambda^\xi \subset \mathbb{Z}_p$, the limiting set also has doubling property. For every $a \in \Lambda^\xi$ and $r > 0$, let $A = B_r(a)$. Then $B_{r/2}(a) \subset A \subset B_r(a)$. For any $y \in \Lambda^\xi \setminus B_r(a)$ and $x \in B_r(a)$, by the strong triangle equality, we have $|x - y|_p = |y - a|_p \geq r > r/2$, namely $\text{dist}(B_r(a), \Lambda^\xi \setminus B_r(a)) \geq r/2$, which shows that Λ^ξ is uniformly disconnected. \square

5. EXAMPLES

In the section we provide certain examples of contractive mappings for which the unconventional limiting set is quasi-symmetrically equivalent to the symbolic Cantor set F^∞ . But, in general, such kind of result is unknown.

Let $f_1(x) = px$ and $f_2(x) = px + 1 - p$. It is clear that f_i is a contractive mapping on \mathbb{Z}_p . For convenience, we denote

$$f_{i_1 i_2 \dots i_n} := f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}$$

and by $x_{i_1 i_2 \dots i_n}$ fixed point of the mapping $f_{i_1 i_2 \dots i_n}$. Note that $x_1 = 0$ and $x_2 = 1$.

For any $x \in \mathbb{Z}_p$ we get

$$f_{i_1 i_2 \dots i_n}(x) = p^n x + (1 - p)(\delta_{2i_1} + \delta_{2i_2}p + \dots + \delta_{2i_n}p^{n-1}),$$

where δ_{2i_k} is a Kroneker symbol. It follows that

$$(5.1) \quad x_{i_1 i_2 \dots i_n} = \frac{\delta_{2i_1} + \delta_{2i_2}p + \dots + \delta_{2i_n}p^{n-1}}{1 + p + p^2 + \dots + p^{n-1}}.$$

Let $\Lambda_0 = \{x_{i_1 i_2 \dots i_n} : \{i_1, i_2, \dots, i_n\} \in \{1, 2\}^n, n \in \mathbb{N}\}$. It is clear that $\Lambda_0 \subset \mathbb{Z}_p$. Note that $x \in \Lambda_0$ if and only if $1 - x \in \Lambda_0$. Denote $\Lambda := \overline{\Lambda_0}$.

Lemma 5.1. *The set Λ has the following properties:*

($\Lambda.1$) *$x \in \Lambda$ if and only if x has the following canonical form*

$$x = p^{\gamma(x)}(1 + x_1 \cdot p + x_2 \cdot p^2 + \dots),$$

where $\gamma(x) \in \mathbb{Z}_+$ and $x_i \in \{0, p - 1\}$, $i = 1, 2, 3, \dots$

($\Lambda.2$) *The set Λ is uniformly perfect.*

Proof. ($\Lambda.1$) From (2.2) one can see that each $x \in \mathbb{Z}_p$ has the following canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 \cdot p + x_2 \cdot p^2 + \dots),$$

where $\gamma(x) \in \mathbb{Z}_+$, $x_0 \neq 0$. Let $x \in \Lambda \setminus \{0\}$.

Since $\Lambda = \overline{\Lambda_0}$ one has $x \in \Lambda$ if and only if there exists a sequence $\{i_k\}_{k \in \mathbb{N}}$, $i_k \in \{1, 2\}$ such that

$$(5.2) \quad \left| p^{\gamma(x)} \left[\sum_{j=0}^n \sum_{i=0}^j x_i p^j + \sum_{j=1}^{\infty} \sum_{i=j}^{n+j} x_i p^{n+j} \right] - \sum_{j=1}^{n+1} \delta_{2i_j} p^{j-1} \right|_p < \frac{1}{p^n}$$

From (5.2) one can see $i_k = 1$, $k = \overline{1, \gamma(x) + 1}$. So, without loss of generality we may assume that $\gamma(x) = 0$.

Let $x \in \Lambda \setminus \{0\}$. Assume that $x_0 \neq 1$. Then for any sequence $\{i_k\}_{k \in \mathbb{N}}$ using non-Archimedean norm's property we get

$$\left| p^{\gamma(x)} \left[\sum_{j=0}^n \sum_{i=0}^j x_i p^j + \sum_{j=1}^{\infty} \sum_{i=j}^{n+j} x_i p^{n+j} \right] - \sum_{j=1}^{n+1} \delta_{2i_j} p^{j-1} \right|_p = |x_0 - \delta_{2i_1}|_p = 1.$$

This contradicts to $x \in \Lambda$. So, we conclude that $x_0 = 1$ if $x \in \Lambda \setminus \{0\}$.

Now we will show that $x_i \in \{0, p-1\}$, $i = 1, 2, 3, \dots$. For a given sequence $\{i_k\}_{k \in \mathbb{N}}$ we define the following sequence

$$a_n(x) = \left| 1 + \sum_{j=1}^n \left(1 + \sum_{i=1}^j x_i \right) p^j + \sum_{j=1}^{\infty} \sum_{i=j}^{n+j} x_i p^{n+j} - \sum_{j=1}^{n+1} \delta_{2i_j} p^{j-1} \right|_p.$$

Note that if $x_1 \notin \{0, p-1\}$ then for any $\{i_k\}_{k \in \mathbb{N}}$ holds

$$a_n(x) = |1 - \delta_{2i_1} + (1 + x_1 - \delta_{2i_2})p|_p = \begin{cases} 1, & \text{if } i_1 = 1 \\ \frac{1}{p}, & \text{if } i_1 = 2. \end{cases}$$

This means that for any sequence $\{i_k\}$ one has $a_n(x) \not\rightarrow 0$ as $n \rightarrow \infty$. This contradicts to $x \in \Lambda_1$.

Now let us assume that $x_i \in \{0, p-1\}$, $i = \overline{1, k}$ and $x_{k+1} \notin \{0, p-1\}$. Redenote elements of $\{1, 2, \dots, k\}$ by $x_{m_1} = x_{m_2} = \dots = x_{m_s} = p-1$ and $x_{m_{s+1}} = x_{m_{s+1}} = \dots = x_{m_k} = 0$. Then we have

$$\begin{aligned} 1 + \sum_{j=1}^{k+1} \left(1 + \sum_{i=1}^j x_i \right) p^j &= 1 + \sum_{j=1}^{m_1-1} p^j + \sum_{j=m_1+1}^{m_2} p^j \\ &+ (2p-1) \sum_{j=m_2}^{m_3-1} p^j + \dots + (sp-s+1) \sum_{j=m_s}^{k-1} p^j \\ &+ (sp-s+1)p^k + (sp-s+1+x_{k+1})p^{k+1} \\ (5.3) \quad &= 1 + \sum_{j \notin \{m_1, m_2, \dots, m_s\}}^{k-1} p^j + p^k + (1+x_{k+1})p^{k+1} + sp^{k+2}. \end{aligned}$$

From (5.3) with the non-Archimedean norm's property, for any sequence $\{i_j\}_{j \in \mathbb{N}}$ one gets

$$a_n(x) \geq \frac{1}{p^{k+1}}.$$

This contradicts to $x \in \Lambda_1$.

Let $x \in \mathbb{Z}_p$ has the following canonical form

$$x = p^{\gamma(x)}(1 + x_1 \cdot p + x_2 \cdot p^2 + \dots),$$

where $\gamma(x) \in \mathbb{Z}_+$ and $x_i \in \{0, p-1\}$, $i = 1, 2, 3, \dots$. For any $n \geq 1$ redenote elements of $\{1, 2, \dots, n\}$ such that $x_{m_1} = x_{m_2} = \dots = x_{m_{s_n}} = p-1$ and $x_{m_{s_n+1}} = x_{m_{s_n+1}} = \dots = x_{m_n} = 0$. Pick a sequence $\{i_k\}_{k \in \mathbb{N}}$ such that $i_k = 1$ if $k \in \{1, 2, \dots, \gamma(x)+1\} \cup \{m_1, m_2, \dots, m_{s_n}\}$ or $k \neq n$ and $i_k = 2$ otherwise. Then for this sequence we get $a_n(x) < p^{-n}$ which means that $x \in \Lambda_1$.

(A.2) We will show that for any $x \in \Lambda_1$ and $r > 0$ there exists $y \in \Lambda_1$ such that

$$\frac{r}{p} \leq |x - y|_p \leq r.$$

According to (A.1) it is enough to check only case $|x|_p > r$. Let us pick a positive integer n such that $p^{-1}r \leq p^{-n} \leq r$. Note that $\gamma(x) < n$. Then take $y \in \Lambda_1$ such that $|y|_p = |x|_p$ and $y_i = x_i$ if $i < n - \gamma(x)$ and $y_{n-\gamma(x)} \neq x_{n-\gamma(x)}$. Thus, we have $|x - y|_p = p^{-n}$. \square

Let $\xi = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$ be a matrix which elements are 1 or 2. For ξ_{ij} we define an action on $\Sigma = \{1, 2\}^{\mathbb{N}}$ as follows

$$(\xi_{ij} * \alpha)_k = \begin{cases} 1, & \text{if } \xi_{ij} + \alpha_k \text{ is odd} \\ 2, & \text{if } \xi_{ij} + \alpha_k \text{ is even.} \end{cases}$$

For $f_1(x) = px$ and $f_2(x) = px + 1 - p$ and a given matrix ξ by Λ^ξ we denote the unconventional limiting set of \mathcal{F}_ξ . We will show that Λ^ξ is quasi-symmetrically equivalent to the symbolic Cantor set F^∞ .

1) Let $\xi = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. In this case one has

$$F_{\alpha,n}(x) = 2 (f_{a_1 a_2 \dots a_n}(x))^2.$$

It follows that

$$\Lambda_0^\xi = \{2x_0^2 : x_0 \in \Lambda_0\}.$$

Hence, $\Lambda^\xi = \{2x^2 : x \in \Lambda\}$.

2) Let $\xi = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$. In this case we get

$$F_{\alpha,n}(x) = 2f_{a_1 a_2 \dots a_n}(x) \cdot f_{\bar{a}_1 \bar{a}_2 \dots \bar{a}_n}(x),$$

where

$$\bar{\alpha}_i = \begin{cases} 1, & \text{if } \alpha_i = 2 \\ 2, & \text{if } \alpha_i = 1. \end{cases}$$

Due to $x_{\bar{\alpha}_1 \bar{\alpha}_2 \dots \bar{\alpha}_n} = 1 - x_{\alpha_1 \alpha_2 \dots \alpha_n}$, one finds

$$\Lambda_0^\xi = \{2x_0(1 - x_0) : x_0 \in \Lambda_0\} \quad \text{and} \quad \Lambda^\xi = \{2x(1 - x) : x \in \Lambda\}.$$

3) Let $\xi = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$. In this case, we get

$$\Lambda_0^\xi = \{x_0^2 + (1 - x_0)^2 : x_0 \in \Lambda_0\} \quad \text{and} \quad \Lambda^\xi = \{x^2 + (1 - x)^2 : x \in \Lambda\}.$$

4) Let $\xi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. In this case, we get

$$\Lambda_0^\xi = \Lambda_0 \quad \text{and} \quad \Lambda^\xi = \Lambda.$$

The uniformly perfectness of the sets Λ^ξ immediately follows from the uniformly perfectness of Λ . According to Theorems 4.8 and 2.1 we conclude that Λ^ξ is quasi-symmetrically equivalent to the symbolic Cantor set F^∞ .

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MUKHAMEDOV FARRUKH, DEPARTMENT OF COMPUTATIONAL & THEORETICAL SCIENCES, FACULTY OF SCIENCE, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. BOX, 141, 25710, KUANTAN, PAHANG, MALAYSIA

E-mail address: far75m@yandex.ru farrukh.m@iiium.edu.my

OTABEK KHAKIMOV, INSTITUTE OF MATHEMATICS, NATIONAL UNIVERSITY OF UZBEKISTAN, 29, DO'RMON YO'LI STR., 100125, TASHKENT, UZBEKISTAN.

E-mail address: hakimovo@mail.ru